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ATTAINABLE KIEFER BOUNDS USING CENSORED SAMPLES FROM LEFT TRUNCATED FAMILY OF DISTRIBUTIONS

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ABSTRACT

We consider left truncated family of distributions in which the densities are in their natural form. Identifying suitable prior densities we compute Kiefer bounds on variance of unbiased estimators of the parametric functions involved in densities. Type II left censored and doubly censored samples are taken into consideration. Further, the bounds are shown to be attained by variances of estimators based on the sample considered. Results are illustrated through examples. The bounds based on complete and censored samples are compared.

KEYWORDS: Censored Samples, Ideal Estimation Equation, Kiefer Bound, Left Truncated Distribution, Minimum Variance Unbiased Estimator, Parametric Function, Variance Bound

1. INTRODUCTION

The information inequality providing lower bound on the variance of estimators due to Fréchet- Cramér-Rao is a land mark in the history of Statistics. In the non- regular family of distributions the lower bounds on the variance such as Fréchet- Cramér-Rao bound, Bhattacharayya bounds cannot be obtained as the support depends upon the parameter and the regularity conditions are violated. The bounds provided by the inequalities due to Chapman and Robbins (1951), Fraser and Guttman (1952), Hammersley (1950), Kiefer (1952), Vincze (1976) can be applied in non-regular situations. Amongst these bounds only the bound due to Kiefer is known to be attained by the variance of UMVU Estimators of the parameter θ in a few situations in non-regular distributions. But it is less familiar and hence less applied. Our efforts are to find its more and more applications. Blischke et.al. (1965-69), Polfeldt (1970), Akahira and Ohyauchi (2007) extended Kiefer's results for asymptotic situations. Bartlett (1982) extended them for parameters of a few more probability distributions. Jadhav and Prasad (1986-87) extended those for some parametric functions in a family of distributions. Jadhav and Shanubhogue (2014) provided attainable Kiefer bounds for left and right truncated distributions and the distributions with both the ends of the support depending on the parameter. All these studies are based either on a small sample or on a large sample. But in the situations such as life testing experiments these results cannot be suitable. In such situations censored samples are chosen using various censoring schemes. Now, we obtain Kiefer bounds on variances of estimators of parametric functions based on censored samples from left truncated distributions.

2. KIEFER BOUND IN TYPE II LEFT CENSORED SAMPLE IN LEFT TRUNCATED FAMILY

We consider a sample in which first (r-1) failures are not observed or are not available. Therefore, observations from r^{th} order statistics onwards only are available. Let the parent population be left truncated r.v. with p.d.f.;

$$f(x;\theta) = \frac{q(x)}{Q(b) - Q(\theta)} \quad ; -\infty < a < \theta < x < b < \infty \tag{2.1}$$

Note that $q(.)$ is a positive real valued function with integration $Q(.)$ so that (2.1) is quite natural. It can be seen that the r[th] order statistics $X_{(r)}$ is complete sufficient for θ.

Let us compute Kiefer bound and assure its attainment through the ideal estimation equation based on generalized difference. We describe the procedure first. The generalized difference of a probability density function $f(x;\theta)$ is defined as

$$\Delta_2 f(x;\theta) = \int f(x;\theta+h)dG_1(h) - \int f(x;\theta+h)dG_2(h) \quad (2.2)$$

where, G_1, G_2 are distribution functions.

Using generalized difference with proper choice of prior distributions Bartlett (1982) writes the ideal estimation equation as

$$\frac{\Delta_2 f(x;\theta)}{f(x;\theta)\Delta_2\theta} = \frac{1}{Var(T)}(T-\theta) \quad (2.3)$$

If $G_2(h) = I_{\{0\}}(x)$, we write this as,

$$\frac{\Delta_1 f(x;\theta)}{f(x;\theta)\Delta_1\theta} = \frac{1}{Var(T)}(T-\theta) \quad (2.4)$$

If such an equation exists, it implies that T is uniformly minimum variance unbiased estimator of θ with its variance attaining its Kiefer bound $K(\theta)$ (UMVUKBE). Then we have the following.

Theorem 2.1

The variance of UMVU estimator of $Q(\theta)$ based on minimum observation in left censored sample from left truncated distribution (2.1) attains Kiefer bound. That is, the estimator is minimum variance unbiased Kiefer bound estimator (UMVUKBE).

Proof

For X with p.d.f. (2.1), $F(x) = \frac{Q(x)-Q(\theta)}{Q(b)-Q(\theta)}$ and $1-F(x) = \frac{Q(b)-Q(x)}{Q(b)-Q(\theta)}$, the p.d.f. of $X_{(r)}$ is given by

$$g_{r:n}(x_{(r)};\theta) = \frac{n!}{(r-1)!(n-r)!}[F(x_{(r)})]^{r-1}[1-F(x_{(r)})]^{n-r} f(x_{(r)}), \;\; \theta < x_{(r)} < b \quad (2.5)$$

$$= \frac{n! \; [Q(x_{(r)})-Q(\theta)]^{r-1}}{(r-1)!(n-r)!}\frac{[Q(b)-Q(x_{(r)})]^{n-r}}{[Q(b)-Q(\theta)]^n} q(x_{(r)}), Q(\theta) < Q(x_{(r)}) < Q(b) \quad (2.6)$$

Let $Q(\theta) = \varphi$. Then the pdf of $X_{(r)}$ can be written as

$$g_{r:n}(x_{(r)};\varphi) = \frac{n![Q(x_{(r)})-\varphi]^{r-1}}{(r-1)!(n-r)!}\frac{[Q(b)-Q(x_{(r)})]^{n-r} q(x_{(r)})}{[Q(b)-\varphi]^n}, \;\; \varphi < Q(x_{(r)}) < Q(b) \quad (2.7)$$

For each fixed $\varphi \in \Phi = \{\varphi; \; g_{r:n}(x_{(r)};\varphi) > 0\} = (0, Q(b))$.

Let $\Phi_\varphi = \{h; (\varphi+h) \in \Phi\} = (-\varphi, Q(b)-\varphi)$ .On subset $(0, Q(b)-\varphi)$ of

$(-\varphi, Q(b)-\varphi)$. let us define prior distributions as

$$dG_1(h) = \frac{[n+1](Q(b)-\varphi-h)^n dh}{[Q(b)-Q(\theta)]^{n+1}}, \;\; 0 < h < Q(b)-\varphi; \; G_2(h) = I_{\{0\}}(h) \quad (2.8)$$

$\Delta_1(Q(b)-\varphi) = E_1(h)$

$= \int_0^{Q(b)-\varphi} h dG_1(h)$

$$= \frac{Q(b)-\varphi}{n+2} \tag{2.9}$$

If φ is incremented to $\varphi + h$, we have,

$$g_{r:n}(x_{(r)};\ \varphi+h) = \frac{n![Q(x_{(r)})-\varphi-h]^{r-1}}{(r-1)!(n-r)!} \frac{[Q(b)-Q(x_{(r)})]^{n-r} q(x_{(r)})}{[Q(b)-\varphi-h]^n} \tag{2.10}$$

and, $\varphi + h < Q(x_{(r)}) < Q(b)$ which implies that $0 < h < Q(x_{(r)}) - \varphi$.

Then we have,

$\int_{\Phi_\varphi} g_{r:n}(x_{(r)};\ \varphi+h) dG_1(h)$

$= \frac{n![Q(b)-Q(x_{(r)})]^{n-r} q(x_{(r)})[n+1]}{(r-1)!(n-r)![Q(b)-Q(\theta)]^{n+1}} \int_0^{Q(x_{(r)})-\varphi} [Q(x_{(r)})-\varphi-h]^{r-1} dh$

$= \frac{n![Q(b)-Q(x_{(r)})]^{n-r} q(x_{(r)})[n+1][Q(x_{(r)})-\varphi]^r}{r!(n-r)![Q(b)-Q(\theta)]^{n+1}}$

$\Delta_1 g_{r:n}(x_{(r)};\ \varphi)$

$= \frac{n![Q(x_{(r)})-\varphi]^r [Q(b)-Q(x_{(r)})]^{n-r} q(x_{(r)})}{(r-1)!(n-r)![Q(b)-Q(\theta)]^n} \left[\frac{(n+r)[Q(x_{(r)})-\varphi]}{r[Q(b)-Q(\theta)]} - 1\right]$

Using this and (2.6), we get,

$\frac{\Delta_1 g_{r:n}(x_{(r)};\varphi)}{g_{r:n}(x_{(r)};\varphi)\Delta_1(Q(b)-\varphi)} = \frac{(n+2)}{[Q(b)-\varphi]}\left[\frac{(n+1)[Q(x_{(r)})-\varphi]-r[Q(b)-\varphi]}{r[Q(b)-\varphi]}\right]$

$$= \frac{(n-r+1)(n+2)}{r[Q(b)-\varphi]^2}\left[\frac{(n+1)Q(x_{(r)})}{(n-r+1)} - \frac{rQ(b)}{(n-r+1)} - \varphi\right] \tag{2.11}$$

The equation (2.11) is the ideal estimation equation which implies that

$\mathrm{T}(x_{(r)}) = \frac{(n+1)Q(x_{(r)})}{(n-r+1)} - \frac{rQ(b)}{(n-r+1)}$ is UMVUE of $\varphi = Q(\theta)$ with its variance given by

$Var[\mathrm{T}(x_{(r)})] = \frac{r[Q(b)-\varphi]^2}{(n-r+1)(n+2)} = \mathrm{K}(\varphi)$, the Kiefer bound on variance of estimator of φ.

Remark 2.1

If the whole sample is observed, $X_{(r)} = X_{(1)}$, Kiefer bounds from complete and censored samples coincide to $\frac{[Q(b)-\varphi]^2}{n(n+2)}$. This is attained by the variance of UMVUE.

Remark 2.2

$\left\{Var[\mathrm{T}(X_{(r)})] = \frac{r[Q(b)-\varphi]^2}{(n-r+1)(n+2)} = \mathrm{K}(\varphi), \mathrm{r} \geq 1\right\}$is an increasing function of r. That is, though the variance of the estimator continues to attain its Kiefer bound it increases with increase in the number of censored observations.

Remark 2.3

The estimators based on left censored samples from left truncated distributions have larger variances

[$\frac{r}{(n-r+1)(n+2)}[Q(b)-\varphi]^2$] than those based on complete sample $\frac{[Q(b)-\varphi]^2}{n(n+2)}$.

$$\frac{\mathrm{Var}\left[\mathrm{T}\left(X_{(r)}\right)\right] \text{ based on left censored sample}}{\mathrm{Var}\left[\mathrm{T}\left(X_{(1)}\right)\right] \text{ based on complete sample}} = \frac{\mathrm{rn}}{\mathrm{n-r+1}} > 1 \text{ if } \mathrm{r,n} > 1.$$

Example 2.1

Let

$f(x;\theta) = e^{-(x-\theta)}, \theta < x < \infty$

Here, $q(x) = e^{-x}, Q(x) = -e^{-x}, Q(b=\infty) = 0, Q(\theta) = -e^{-\theta}$ $.F(x) = 1 - e^{-\theta}$. Here, Kiefer bound on the variance of u-estimator of $e^{-\theta}$ based on

(i) complete sample is $\frac{\mathrm{e}^{-2\theta}}{\mathrm{n(n+2)}}$ and (ii) censored sample is $\frac{re^{-2\theta}}{(n-r+1)(n+2)}, 1 < r < n.$

Example 2.2

Let

$f(x;\theta) = (b-\theta)^{-1}, \theta < x < b$

Here, $(x) = 1.\ Q(x) = x$. Then, Kiefer bound on the variance of u-estimator of θ based on

(i) complete sample is $\frac{(b-\theta)^2}{n(n+2)}$ and (ii) censored sample is $\frac{r[b-\theta]^2}{(n-r+1)(n+2)}; 1 \le r \le n.$

3. KIEFER BOUND FROM DOUBLY CENSORED SAMPLE FROM LEFT TRUNCATED FAMILY

In life testing experiments some initial and last failures are not observed. Thus, we have ordered observations, say, from $X_{(r)}$ to $X_{(s)}$; $r < s$ which is doubly censored sample. Now, we shall obtain Kiefer bound based on this sample.

Theorem 3.1

Doubly censored sample on variable having left truncated probability density function (pdf) provides UMVU estimator of function of truncation parameter involved in pdf with its variance attaining Kiefer bound.

Proof

Let the pdf of the variable be from left truncated family as in (2.1). Consider a doubly censored sample in which only the observations from rth order statistic $X_{(r)}$ to sth order statistic $X_{(s)}$ are observed. Then the likelihood function of these observations is as below.

$$L(\underline{x}|\theta) = \frac{n!}{(r-1)!(n-s)!}[F(x_{(r)})]^{r-1}[1-F(x_{(s)})]^{n-s} \prod_{i=r}^{s} f(x_{(i)}) \tag{3.1}$$

Therefore, by factorization theorem, $X_{(r)}$ is sufficient statistic for θ.The pdf of $x_{(r)}$ is given by,

$$f_{r:n}(x_{(r)};\theta) = \frac{n!}{(r-1)!(n-r)!}\left[F(x_{(r)})\right]^{r-1}\left[1-F(x_{(r)})\right]^{n-r} f(x_{(r)}),\ \theta < x_{(r)} < b$$

$$= \frac{n!}{(r-1)!(n-r)!} \frac{[Q(x_{(r)}) - Q(\theta)]^{r-1}}{[Q(b) - Q(\theta)]^{n}} \left[Q(b) - Q(x_{(r)})\right]^{n-r} q(x_{(r)}). \tag{3.2}$$

Equation (3.2) can be written, by putting, $Q(\theta) = \varphi$ as;

$$g_{r:n}(x_{(r)};\theta) = \frac{n!}{(r-1)!(n-r)!} \frac{[Q(x_{(r)}) - \varphi]^{r-1}}{[Q(b) - \varphi]^{n}} \left[Q(b) - Q(x_{(r)})\right]^{n-r} q(x_{(r)}) \tag{3.3}$$

Then, the results follow from (3.3), (2.6) and (2.8).

Remark 3.1

The results from type II left censored and doubly censored sample in left truncated family of probability distributions is the same.

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